from which $\overline{m} = m_0 - 0{\cdot}855$ and $\sigma_m = 0{\cdot}731$ are deducible, values which agree with those found by Prof. Pearson.

5. It does not seem to me that Burnham's catalogue of double stars can be used as a basis for a discussion of the departure of the actual distribution of the stars from a random distribution. As far as magnitude 9·5, Hussey and Aitken have made systematic search for all double stars. Beyond this magnitude no systematic search has been made. Further, the definition of a double star is a very elastic one. Thus on page 1 of Burnham's catalogue, α Andromedæ (2·0) with a companion of magnitude 11·2 at a distance of 65″ is counted as a double star. Two faint stars of magnitude 9·0 or 10·0 at this distance would not be classified as double, and the photographs taken for the Astrographic Catalogue contain many such stars.

The Conditions Necessary for Discontinuous Motion in Gases.

By G. I. Taylor, B.A., Trinity College, Cambridge.

(Communicated by Prof. Sir Joseph Larmor, Sec. R.S. Received July 11, 1910.)

The possibility of the propagation of a surface of discontinuity in a gas was first considered by Stokes[*] in his paper "On a Difficulty in the Theory of Sound." This paper begins with a physical interpretation of Poisson's integral of the equation of motion of a gas in one dimension. The integral in question is $w = f\{z-(a+w)t\}$; and it represents a disturbance of finite amplitude moving in a gas for which the velocity of propagation of an infinitesimal disturbance is a; w is the velocity of the gas in the direction of the axis z. It is shown that the parts of the waves in which the velocity of the gas is w travel forward with a velocity $a+w$, and that there is in consequence a tendency for the crests to catch up the troughs. After a certain time, and at a certain point in space, the value of $\partial w/\partial z$ will become negatively infinite; a discontinuity will then occur, and Poisson's integral will cease to apply.

Stokes then leaves the subject of oscillatory waves and proceeds to consider whether it is possible to maintain a sharp discontinuity in a gas which obeys Boyle's law ($p = a^2\rho$). His argument, slightly modified by Lord Rayleigh, is as follows:—

Suppose that a travelling discontinuity can exist. Give the whole gas

[*] 'Phil. Mag.,' 1848, vol. 33, p. 349; 'Collected Papers,' vol. 1.

such a motion that the discontinuity is brought to rest. Consider then a gas which is moving with uniform velocity u_1 up to a discontinuity. At this point the velocity suddenly changes to u_2; and the gas moves on uniformly at this speed. Let ρ_1 and ρ_2 be the corresponding densities, p_1 and p_2 the corresponding pressures.

The equation of continuity of mass is

$$\rho_1 u_1 = \rho_2 u_2. \tag{1}$$

The equation of conservation of momentum is

$$p_1 - p_2 = a^2\rho_1 - a^2\rho_2 = \rho_1 u_1 (u_1 - u_2). \tag{2}$$

If u_1 and ρ_1 be given, these two equations determine u_2 and ρ_2.

Against this theory, however, Lord Rayleigh* raised the objection that the equation of energy, $\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 = a^2 \log \rho_1 - a^2 \log \rho_2$, cannot, in general, be satisfied simultaneously with (1) and (2).

In a recent note† he adds a remark that it is possible that energy might be lost at the discontinuity, but it cannot be supposed that energy is gained. Lord Rayleigh further points out that the energy lost must be converted into heat, and that this complication must be taken into account. This has been done by C. V. Burton‡ and by Hugoniot,§ but their equations have the same disadvantage as those of Stokes, in that they contain no indication that the motion represented by them is irreversible.

In the case considered by Stokes it is evident that the motion is irreversible; in fact, it is only the front of a compression that can possibly travel unchanged. For, if for an instant the sharp discontinuity were to disappear, leaving a small transition layer in which the velocity might vary continuously from u_1 to u_2, then the back part of the layer would travel forward relatively to the front part with a velocity $u_1 - u_2$. Hence if u_1 exceeds u_2 any such transition layer will become obliterated owing to the greater velocity behind, and the discontinuity will thus be maintained. This is the case of the front of a wave of condensation. If, however, the wave is a wave of rarefaction, that is, if u_1 is less than u_2, then the layer of transition will get wider, and the sharp discontinuity will not be re-established.

The object of this paper is to discuss in detail what actually does occur at a discontinuity, and to determine, in the general case of a gas whose characteristics are known, whether a discontinuity obtained by the method Stokes is a physically possible feature.

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* 'Theory of Sound,' vol. 2, p. 41.

† 'Roy. Soc. Proc.,' A, 1908, vol. 81, p. 449.

‡ 'Phil. Mag.,' 1893, vol. 35, p. 317.

§ See Lamb's 'Hydrodynamics,' note on p. 466, 3rd edition

It is evident that a plane of absolutely sharp or mathematical discontinuity cannot occur in any real gas. When, owing to change of type, there is a sudden compression or rarefaction of the material in crossing any boundary, modified physical laws must come into operation whose effect is to prevent abrupt discontinuity from being formed. Some clue to the nature of the processes involved in this case is afforded by the kinetic theory of gases; for when the change in velocity is very sudden, the molecules which are moving faster will penetrate among those which are moving more slowly, and an irreversible redistribution of velocities will ensue. This suggests that heat conduction and viscosity are, in the case of a real gas, the causes of the production of dissipative heat; it will be shown that under certain conditions they are also sufficient to produce permanence of type in the layer of transition.

Consider a continuous disturbance of permanent type in a gas whose characteristic equations are known. Give the whole system such a velocity that the disturbance is brought to rest; the motion is then steady.

Let A and B be two planes which move with the gas, and let p', ρ', u', E′, and p, ρ, u, E, be the pressure, density, velocity, and internal energy of unit mass of the gas at A and B respectively. Since θ, the temperature, is a function of p and ρ, and E is a known function of p, ρ, and θ, therefore E may be regarded as a function of independent variables p and ρ.

The equation of continuity of matter is

$$\rho' u' = \rho u = \omega. \tag{3}$$

The rate of gain of momentum between A and B is $\omega\,(u-u')$. The equation of momentum for the gas between A and B is therefore

$$(p+\mathrm{X})+\omega u = (p'+\mathrm{X}')+\omega u', \tag{4}$$

where X and X′ are the viscous normal forces which act over the planes B and A respectively.

The work done on the gas between A and B in a small interval of time dt is

$$(p'+\mathrm{X}')\,u'dt-(p+\mathrm{X})\,udt.$$

The increase of its kinetic energy in time dt is

$$\tfrac{1}{2}\,\omega\,(u^2-u'^2)\,dt.$$

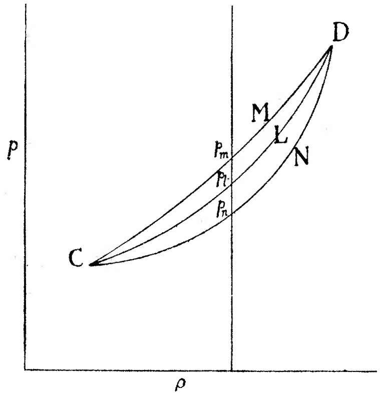
The increase in internal energy in the same time is $(\mathrm{E}-\mathrm{E}')\,\omega\,dt$.

The amount of heat conducted away from the mass of gas between A and B in time dt is $\omega\,(\xi - \xi')\,dt$, where $\omega\xi$ and $\omega\xi'$ are the rates at which heat, measured in mechanical units, is conducted across the planes B and A.

The equation of energy for the gas between A and B is therefore

$$(p+\mathrm{X})\,u+\tfrac{1}{2}\,\omega u^2+\omega\,(\mathrm{E}+\xi) = (p'+\mathrm{X}')\,u'+\tfrac{1}{2}\,\omega u'^2+\omega\,(\mathrm{E}'+\xi'). \tag{5}$$

Since E is a function of p and ρ and $u = \omega/\rho$, the state of the gas at any time may be completely represented by a point in a plane diagram such as is annexed; pressure is represented by the ordinates and density by the abscissæ.



If C and D are the points which represent the state of the gas at the two ends of the transition layer in which the velocity changes from u_1 to u_2, then the state of the gas along that layer are represented by the points on some curved line L joining C and D. It will be possible by means of equations (4) and (5) to determine values of X and ξ so that any given line joining C and D may represent the state of the gas in the transition layer; but the motion so represented will not be thermodynamically possible unless the coefficients of conduction and viscosity are both positive. If x represents distance in the direction in which the gas is travelling, these conditions become

$$\text{X and } \partial u/\partial x \text{ must have opposite signs,}$$

$$\xi \text{ and } \partial \theta/\partial x \text{ must have opposite signs.}$$

Hence at the front of a condensation X is positive and ξ is negative while for a rarefaction X is negative and ξ is positive.

Construct curves M and N to represent the relations obtaining between p and ρ when $\mathrm{X} = 0$ and when $\xi = 0$ respectively. Let p_m, p, and p_n represent the pressures at points on M, L, and N corresponding to a particular value ρ of the density.

The equation to M is obtained by dropping X from (4), and is

$$p_m + \omega u = p_1 + \omega u_1,$$

and since (4) gives

$$(p + \mathrm{X}) + \omega u = p_1 + \omega u_1,$$

therefore

$$p_m = p + \mathrm{X}.$$

Similarly it can be shown that $\mathrm{E}_n = \mathrm{E} + \xi$.

In a condensation therefore $p_m > p$ and $E_n < E$, and in a rarefaction $p_m < p$ and $E_n > E$.

Also $E_n - E = (\theta_n - \theta) C_v$ where C_v is the specific heat at constant volume; and $p_n - p$ must be of the same sign as $\theta_n - \theta$; hence in a condensation $p_m > p > p_n$, and in a rarefaction $p_m < p < p_\omega$.

Now the equations to the lines M and N depend only on the relations which exist between pressure, density, temperature, and internal energy, that is on the characteristic equations of the gas, and not at all on its viscosity or its conductivity; for if either X or ξ is small the other can be eliminated.

Hence if a discontinuity is specified by the equations

$$u_1 \rho_1 = u_2 \rho_2 = \omega,$$

$$p_1 + \omega u_1 = p_2 + \omega u_2,$$

$$p_1 u_1 + \tfrac{1}{2} \omega u_1^2 + \omega E_1 = p_2 u_2 + \tfrac{1}{2} \omega u_2^2 + \omega E_2,$$

connecting the two uniform states between which it lies, and it is desired to find out whether it is thermodynamically possible, draw the lines M and N joining C and D, which are the points representing the states of the gas on the two sides of the discontinuity. If the line M lies above the line N (see diagram) so that p_m is greater than p_n then a condensation is possible. If the line M lies below the line N a rarefaction is possible. If the line M cuts the line N at any point between C and D, neither is possible.

The only special case of any importance is that of a perfect gas whose characteristic equations are

$$\frac{p}{\rho} = R\theta, \qquad E - E_1 = \frac{R}{\gamma - 1} (\theta - \theta_1),$$

where γ is the ratio of the specific heats.

The general criterion is as above; but if we also assume constant conductivity κ and viscosity μ, the circumstances can be followed out in detail. In this case it may be shown that

$$X = -\frac{4}{3} \frac{\mu}{\omega} \frac{du}{dx}^* \quad \text{and} \quad \omega\xi = -J\kappa \frac{d\theta}{dx}.$$

Substituting these values in equations (4) and (5),

$$p - \frac{4}{3} \frac{\mu}{\omega} \frac{du}{dx} + \omega u = p_1 + \omega u_1,$$

and
$$\left(p - \frac{4}{3} \frac{\mu}{\omega} \frac{du}{dx}\right) u + \omega \frac{u^2}{2} + \omega E - J\kappa \frac{d\theta}{dx} = p_1 u_1 + \omega \frac{u_1^2}{2} + \omega E_1.$$

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* See Rayleigh's 'Theory of Sound,' vol. 2, p. 315.

From these two equations, together with the equations

$$\frac{p}{\rho} = \mathrm{R}\theta, \qquad u\rho = \omega, \qquad \mathrm{E}-\mathrm{E}_1 = \frac{\mathrm{R}}{\gamma-1}(\theta-\theta_1),$$

the quantities p, ρ, E and θ may be eliminated.

The resulting equation is

$$\omega \mathrm{B}u - \tfrac{1}{2}\,\omega u^2 + \frac{\omega}{\gamma-1}\left(\frac{4\mu}{3\omega}u\frac{du}{dx} + \mathrm{B}u - u^2\right) - \frac{\kappa \mathrm{J}}{\mathrm{R}}\left\{\frac{4\mu}{3\omega}\frac{d}{du}\left(u\frac{du}{dx}\right) + (\mathrm{B}-2u)\frac{du}{dx}\right\} = \omega \mathrm{A},$$

where $\omega\mathrm{B} = p_1 + \omega u_1$ and $\omega\mathrm{A} = p_1 u_1 + \frac{1}{2}\omega u_1{}^2 + \omega\mathrm{E}_1$.

It may be written in the simplified form

$$y\,\{\mathrm{P}\,dy/du - \mathrm{Q}u + \mathrm{S}\} + \mathrm{M}u^2 - \mathrm{N}u + \mathrm{A} = 0, \tag{6}$$

where $y = u\dfrac{du}{dx}$, $\quad \mathrm{P} = \dfrac{4\mu\kappa\mathrm{J}}{3\omega^2\mathrm{R}}$, $\quad \mathrm{Q} = \dfrac{2\kappa\mathrm{J}}{\mathrm{R}\omega} + \dfrac{4\mu}{3\omega(\gamma-1)}$,

$$\mathrm{S} = \frac{\kappa\mathrm{BJ}}{\mathrm{R}\omega}, \qquad \mathrm{M} = \frac{\gamma+1}{2(\gamma-1)}, \qquad \mathrm{N} = \frac{\mathrm{B}\gamma}{\gamma-1}.$$

If either $\kappa = 0$ or $\mu = 0$ (6) may be solved in the form

$$x = \mathrm{A}\log(u_1-u) + \mathrm{B}\log(u-u_2),$$

where u_1 and u_2 are the roots of $\mathrm{M}u^2 - \mathrm{N}u + \mathrm{A} = 0$.

If, however, neither μ nor κ vanish, (6) cannot be solved in finite terms, but if $u_1 - u_2$ be small compared with u_1 an approximate solution can be obtained.

If u_1 be the greater of the two roots of $\mathrm{M}u^2 - \mathrm{N}u + \mathrm{A} = 0$, the solution is

$$x = \frac{\mathrm{Q}u_1 - \mathrm{S}}{\mathrm{M}(u_1-u_2)}\log\frac{u_1-u}{u-u_2}. \tag{7}$$

By substituting for Q and S their values and remembering that $u_1 - u_2$ is small compared with u_1, it may be shown that

$$\mathrm{Q}u_1 - \mathrm{S} = \frac{\kappa\mathrm{J}}{\mathrm{R}\omega}\left(1-\frac{1}{\gamma}\right) + \frac{4\mu u_1}{3\omega(\gamma-1)},$$

which is positive. Hence (7) represents a condensation; for when u approaches u_1, x approaches $-\infty$, and when u approaches u_2, x approaches $+\infty$. From (7) it is possible to calculate approximately the thickness of the transition layer. It is evident that the distance between the planes where the velocities are u_1 and u_2, is infinite; but to obtain some idea of the extent of the transition layer consider the thickness T of the layer in which the velocity changes from $\frac{9}{10}u_1 + \frac{1}{10}u_2$ to $\frac{1}{10}u_1 + \frac{9}{10}u_2$. Substituting these values for u in (7)

$$T = \frac{Qu_1 - S}{M(u_1 - u_2)} \cdot 2 \log_e 9,$$

and inserting the values of γ, ρ, μ, and κ for air,

$$\gamma = 1{\cdot}4, \qquad \rho = 1{\cdot}3 \times 10^{-3}, \qquad \mu = 1{\cdot}9 \times 10^{-4}, \qquad \kappa = \frac{1{\cdot}6\,\mu R}{J(\gamma - 1)},^{*}$$

there is obtained $(Qu_1 - S)/M = 0{\cdot}22$ and $T = (u_1 - u_2)^{-1}$ approximately.

In the case of waves of percussion it is known that the velocity differs appreciably from that of sound. In that case $u_1 - u_2$ would be considerable and its reciprocal would be small, so that the motion would closely approximate to an abrupt discontinuous one. In the case of ordinary sounds, however, the relative velocities of air in different parts of a wave are small, so that T would be large compared with a wave-length, and nothing in the nature of a sharp discontinuity would ever be established.

On the Radium Content of Basalt.

By the Hon. R. J. Strutt, F.R.S., Professor of Physics, Imperial College
of Science, South Kensington.

(Received July 16, 1910.)

In a former paper† I gave measurements of the amount of radium in representative igneous rocks. In the reduction of these measurements a value of the equilibrium ratio between radium and uranium given by Rutherford and Boltwood was used, which has subsequently been corrected by those authors. My results were reprinted with the necessary amendment by Eve and McIntosh.‡

Subsequent to the publication of my first paper other experimenters have made similar measurements, with results in most cases substantially the same.§

Prof. J. Joly, however, has arrived at values considerably higher. His results are most conveniently referred to in his book "Radioactivity and

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[*] O. E. Meyer, 'Kinetic Theory of Gases,' English edition, p. 292.

[†] 'Roy. Soc. Proc.,' A, vol. 77, p. 472.

[‡] 'Phil. Mag.,' August, 1907, p. 231.

[§] See Eve and McIntosh, *loc. cit.* Farr and Florance, 'Phil. Mag.,' November, 1909, p. 812. Schlundt and Moore, 'U.S. Geol. Survey Bull.,' vol. 395, p. 26. Fletcher, 'Phil. Mag.,' July, 1910, p. 36.